

Consequences of Binary Necklaces

Abstract

A binary necklace is a circular arrangement of zeros and ones, considered up to rotation. Applying exclusive OR to neighboring bits turns these objects into a finite dynamical system. This introduction develops the resulting connections among power sets, group actions, parity, directed graphs, and periodic patterns. Worked examples show how rotation classes and graph components account for every binary word, why different components can share a graph shape, and how a repeating necklace can continue to rotate in a fixed coordinate system. Elementary proofs explain the role of odd lengths and powers of two. The final section connects these ideas to an algebraic classification. Familiarity with sets, functions, and arithmetic modulo two is sufficient for the main exposition.

1 From subsets to circular words

Consider a ring of n positions, each carrying either 0 or 1. Reading from a chosen position produces a binary word. Moving the starting position changes the written word without changing the circular arrangement. The standard mathematical object that forgets the starting position is a *binary necklace*. Throughout this paper, the orientation around the circle is retained; reflection will be considered separately.

1.1 A word records a subset

Let $X_n = \mathbb{Z}/n\mathbb{Z}$, whose elements are the positions $0, 1, \dots, n-1$ in cyclic order. A word $w = w_0w_1 \cdots w_{n-1}$ records the subset

$$A_w = \{i \in X_n : w_i = 1\}.$$

For instance, at $n = 4$, the word 0101 records $\{1, 3\}$. Every subset has one characteristic word and every word records one subset. Thus

$$\mathcal{P}(X_n) \longleftrightarrow \{0, 1\}^n, \quad |\mathcal{P}(X_n)| = 2^n.$$

The cyclic order is additional structure on the ground set. Relabeling one cyclically ordered set as another gives an isomorphic construction.

1.2 A necklace is a rotation orbit

Fix the right rotation

$$R(w_0w_1 \cdots w_{n-1}) = w_{n-1}w_0 \cdots w_{n-2}. \quad (1.1)$$

Since $R^n = I$, its powers form the cyclic group C_n . This group acts on binary words by rotation. The *orbit* of w is

$$[w] = \{R^j w : 0 \leq j < n\}. \quad (1.2)$$

This set is the necklace represented by w . Brackets in this paper denote the whole rotation class, not a word with distinguished endpoints. For example,

$$[0001] = \{0001, 1000, 0100, 0010\}, \quad [0101] = \{0101, 1010\}.$$

The words 0000 and 1111 each have an orbit of size one. Necklaces of length n therefore need not contain n distinct words.

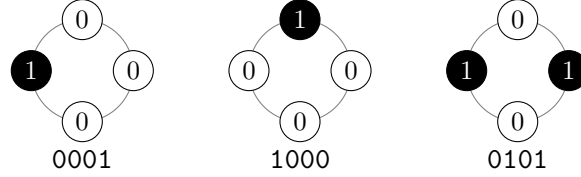


Figure 1: Reading clockwise from the top gives the displayed words. The first two drawings represent the same necklace. The alternating necklace on the right has only two distinct rotations.

Two words are *rotation-equivalent* if one is a rotation of the other. This is an equivalence relation: every word is equivalent to itself, the relation is symmetric, and it is transitive. Its classes partition all 2^n words. The set of necklaces is the *orbit space* $\mathcal{N}_n = \{0, 1\}^n / C_n$.

1.3 Least period and primitive necklaces

The *least spatial period* of w is the smallest positive integer d such that $R^d w = w$. It divides n , and w is a repetition of a word of length d . This shortest block is a *primitive root* of the written word. Changing the starting position rotates that root, so its necklace is well defined independently of the representative.

For example, 001001001001 has least period 3 and primitive root 001. Its necklace consists of three distinct words. The word 001001 is another repeated word; its length remains 6 even though its least period is 3.

Proposition 1.1. *A word of least spatial period d has exactly d distinct rotations.*

Proof. The words $w, R w, \dots, R^{d-1} w$ are distinct by minimality of d , and further rotations repeat them. Equivalently, the stabilizer of w in C_n has n/d elements, so the orbit-stabilizer theorem gives $|[w]| = n/(n/d) = d$. $\square$

A necklace of length n is *primitive*, or *aperiodic*, if its least spatial period is n . Here “aperiodic” means that no smaller rotational period exists. It says nothing about the temporal evolution introduced below.

2 Counting the necklaces

At length 4 the six rotation classes are represented by

$$0000, 0001, 0011, 0101, 0111, 1111.$$

Their respective sizes are 1, 4, 4, 2, 4, 1, which sum to 16. Simply dividing 2^4 by 4 would be incorrect because some orbits are shorter than others.

2.1 Averaging fixed points

Burnside's lemma counts orbits by averaging the numbers of objects fixed by the group elements. A rotation through k positions splits the n positions into $\gcd(n, k)$ cycles. A word fixed by that rotation must be constant on each such cycle, giving $2^{\gcd(n, k)}$ choices. Therefore the number of binary necklaces of length n is

$$N(n) = \frac{1}{n} \sum_{k=0}^{n-1} 2^{\gcd(n, k)} = \frac{1}{n} \sum_{d|n} \varphi(d) 2^{n/d}, \quad (2.1)$$

where $\gcd(n, 0) = n$ and $\varphi(d)$ is Euler's totient function, the number of integers from 1 to d relatively prime to d .

The averaging can be justified directly. Count pairs (w, R^k) for which $R^k w = w$. An orbit of size d contains d words, each fixed by n/d rotations, so every orbit contributes exactly n pairs. The first sum in (2.1) counts the same pairs by rotation. The second groups rotations by their orders. At $n = 4$ this gives

$$N(4) = \frac{16 + 2 + 4 + 2}{4} = 6.$$

2.2 A consequence for primes

If p is prime, a length- p word has least period 1 or p . The only period-one words are $\mathbf{0}$ and $\mathbf{1}$, the constant-zero and constant-one words. The other $2^p - 2$ words occur in rotation classes of size p . Hence

$$p \mid 2^p - 2.$$

This is the base-two case of Fermat's little theorem. The proof is a partition argument: divisibility follows because a set has been divided into equal-sized classes.

3 Adjacent XOR and the binary Ducci map

Addition modulo two has the table

+	0	1
0	0	1
1	1	0

and is also called *exclusive OR*, or XOR. Applying it coordinate by coordinate turns binary words into the vector space $V_n = \mathbb{F}_2^n$, where $\mathbb{F}_2$ is the field with two elements. In subset language, this addition is symmetric difference: $A_{u+v} = A_u \triangle A_v$.

Define the *adjacent-XOR map*

$$D(w) = w + Rw, \quad (Dw)_i = w_i + w_{i-1}, \quad (3.1)$$

with indices modulo n . Thus each new bit records whether a pair of neighboring bits differs. For example,

$$\begin{array}{r} w \quad 0001 \\ Rw \quad 1000 \\ \hline Dw \quad 1001 \end{array} \quad \text{and hence} \quad \overline{D}[0001] = [0011].$$

The notation $\overline{D}$ denotes the induced operation on necklaces.

This map is the *binary Ducci map*. The usual Ducci operation takes absolute differences of cyclically adjacent entries; on bits, $|a - b| = a + b \pmod{2}$. Choosing the other neighbor reverses the spatial convention. It gives $I + R^{-1} = R^{-1}(I + R)$, which induces exactly the same map on rotation classes. Thus the necklace operation is independent of this choice of direction. “Adjacent XOR” describes the rule; “binary Ducci map” places it in its established mathematical context.

3.1 Why the operation is well defined

A definition using a representative must give the same necklace no matter which rotation is chosen. This follows from

$$D(R^j w) = R^j D(w). \quad (3.2)$$

Indeed, $D = I + R$ commutes with every power of R . Consequently $\overline{D}([w]) = [Dw]$ is well defined. This is an example of an *equivariant* map descending to an orbit space.

Only *adjacent* rotations define this particular rule. Replacing R by R^k would in general give a different map. Also, the orbit space of necklaces is not itself a vector space under an arbitrary choice of representatives. At length 3, for example, both 001 and 010 represent the same necklace, but

$$001 + 001 = 000, \quad 010 + 001 = 011.$$

The output necklaces differ. The linear algebra takes place on words; equivariance is what permits this particular map on necklaces.

3.2 Repetition respects the dynamics

If $d \mid n$, repeat a length- d word n/d times to obtain a length- n word. Every adjacent pair in the repeated word is the corresponding pair in the shorter circular word, including at the joins. Repetition therefore commutes with R and D . A shorter necklace and its repetitions have compatible trajectories, although they belong to different spaces $\mathcal{N}_d$ and $\mathcal{N}_n$.

More generally, the least spatial period cannot increase under D : if $R^d w = w$, then $R^d Dw = Dw$. The output’s least period divides d . It may decrease, as in $0101 \mapsto 1111$.

4 Predecessors, parity, and complementation

The *Hamming weight* $\text{wt}(w)$ is the number of ones in a word. Its parity is $\sum_i w_i$ in $\mathbb{F}_2$. Every adjacent-XOR output has even weight, because each input bit contributes to two output positions:

$$\sum_i (Dw)_i = \sum_i (w_i + w_{i-1}) = 0. \quad (4.1)$$

There is also a geometric interpretation. Ones in Dw mark changes between neighboring bits. Traveling once around a circle, the number of changes from zero to one must equal the number from one to zero.

Proposition 4.1 (The preimage criterion). *A word y has a preimage under D if and only if $\text{wt}(y)$ is even. When preimages exist, there are exactly two, and they are complements.*

Proof. Necessity follows from (4.1). Conversely, choose w_0 arbitrarily and use $w_i = y_i + w_{i-1}$ for $1 \leq i < n$. The remaining equation $w_0 + w_{n-1} = y_0$ holds precisely when $\sum_i y_i = 0$. Thus either choice of w_0 gives a solution. Changing that choice changes every bit, so the two solutions are complements. $\square$

Write $\bar{w} = w + \mathbf{1}$ for the *complement*. Since $D\mathbf{1} = \mathbf{0}$, one has $D\bar{w} = Dw$. Complement is different from additive inverse: in $\mathbb{F}_2^n$, every word is its own additive inverse. The kernel of D is $\{\mathbf{0}, \mathbf{1}\}$, and its image is the $(n - 1)$ -dimensional subspace of even-weight words.

On necklaces, an even-weight output has one or two distinct predecessor vertices. To see why there are no more, rotate any predecessor so that its image is a chosen representative y , and apply Proposition 4.1. The two word solutions may lie in the same necklace. For example,

$$D(0101) = D(1010) = 1111, \quad [0101] = [1010].$$

Thus $[1111]$ has just one predecessor necklace. By contrast, the two predecessors of $[0011]$ are $[0001]$ and $[0111]$. An odd-weight necklace has none. The term *preimage* describes a relation, not a single inverse function defined everywhere.

4.1 Self-complementary necklaces

A necklace is *self-complementary* if $[\bar{w}] = [w]$: some rotation exchanges all zeros with ones. For a word of least spatial period d , this is equivalent to

$$d \text{ even}, \quad R^{d/2}w = \bar{w}. \quad (4.2)$$

Indeed, if $R^s w = \bar{w}$, then $R^{2s} w = w$ but $R^s w \neq w$. In the cyclic group of shifts modulo d , the only element of order two is the half-turn $d/2$. Conversely, that half-turn establishes self-complementarity.

The primitive block therefore has complementary halves. The full length- n word need not: 0101 is self-complementary, but its two length-two halves are both 01 . Its primitive block 01 has the complementary halves 0 and 1 .

5 From trajectories to functional graphs

Draw one vertex for each necklace of length n , and one directed edge $[w] \rightarrow [Dw]$. The result is a *functional digraph*, denoted $\mathcal{Q}_n$. Every vertex has exactly one outgoing edge, although it may have zero, one, or two incoming edges. Self-loops are retained. The edges record adjacent XOR on subsets after rotations have been identified; they do not describe subset inclusion.

Starting at a vertex and following its outgoing edges gives its *forward trajectory*. Because the graph is finite, some vertex eventually repeats, after which the trajectory follows a directed cycle. The earlier vertices form the transient part of that trajectory.

A *weakly connected component* is a maximal set of vertices connected when edge directions are ignored. Each such component has exactly one directed cycle: an edge joins two vertices with the same eventual cycle, so two different eventual cycles cannot lie in one component. The remaining vertices form rooted trees directed toward the cycle. Such a component is also the *basin of attraction* of its cycle. It usually branches and is not merely a path.

The *cycle length* is the number of necklaces on the cycle. The *depth* of a transient vertex is the number of steps needed to reach the cycle; cycle vertices have depth zero. A vertex with no predecessors is a leaf of an incoming tree. At the cycle root, the incoming cycle edge is distinct from any incoming tree edges.

5.1 Two components with the same shape

At $n = 3$, all four necklaces and their images are as follows. Both components consist of one looped

Necklace	Number of represented words	Image	Component
[000]	1	[000]	A
[111]	1	[000]	A
[011]	3	[011]	B
[001]	3	[011]	B

Table 1: The complete necklace dynamics at length 3.

root with one leaf pointing to it. They are *isomorphic*: a bijection between their vertices preserves the directed edges. There are therefore two components but only one *graph type*.

Nevertheless, component A represents two words, while component B represents six. To retain this information, give a necklace vertex the weight $\|w\|$, its number of represented words, and define the *mass* of a component C by

$$\|C\| = \sum_{[w] \in V(C)} \|w\|. \quad (5.1)$$

Unweighted graph isomorphism need not preserve mass.

5.2 A complete branching example

At $n = 4$, all six necklaces lie in the component shown in Figure 2. The zero necklace is its only cycle vertex. Its two depth-four leaves are distinct, even though their subsequent trajectories agree.

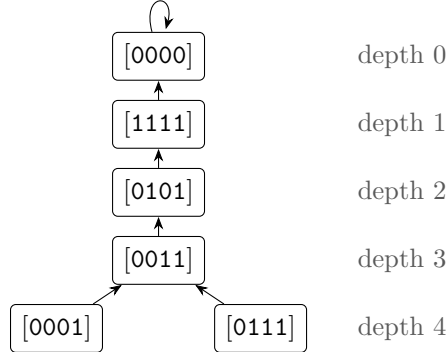


Figure 2: The entire graph $\mathcal{Q}_4$. The vertex weights are 1, 1, 2, 4, 4, 4, from the root downward, summing to 16.

Reading total indegrees by depth gives the branching sequence

$$2, 1, 1, 2, 0.$$

The first 2 counts the root's loop and its incoming tree edge. The last 0 records the leaves. For a tree whose branching is uniform at each depth, these numbers describe its shape. Such uniformity is additional structure; it does not hold for every functional graph. One can encode the example as G_1 -21120, where the subscript is the cycle length and the symbols form a word over $\{0, 1, 2\}$.

5.3 Accounting for the power set

Every word belongs to one rotation class, and every necklace belongs to one graph component. Taking the union of the rotation classes in each component therefore partitions $\mathcal{P}(X_n)$. In particular,

$$\sum_{C \text{ component of } \mathcal{Q}_n} \|C\| = 2^n. \quad (5.2)$$

Grouping isomorphic components is a further classification, not an identification of their subsets. A complete description should retain the graph types, their component multiplicities, and the represented word counts. These are three different pieces of information.

6 Why odd lengths and powers of two behave differently

The simplest structural results already distinguish two large families of lengths. They follow from the same linear map $D = I + R$.

6.1 Odd lengths: cycles with one leaf at each vertex

Proposition 6.1. *If n is odd, every even-weight word is periodic under D , and every odd-weight word enters the periodic set after one step. Every component of $\mathcal{Q}_n$ is a directed cycle with one leaf attached to each cycle vertex.*

Proof. Let E be the subspace of even-weight words. It is preserved by D . The kernel of D consists of $\mathbf{0}$ and $\mathbf{1}$, but $\mathbf{1}$ has odd weight. Thus the restriction $D : E \rightarrow E$ is injective, and hence bijective because E is finite. Every element of E is consequently periodic.

An even-weight word has two complementary preimages, exactly one even and one odd. The even predecessor lies on its periodic orbit. The odd predecessor has no predecessors itself. This remains true after taking rotation classes: parity is preserved by rotation, and an equivariant bijection on E gives a bijection on its necklaces. $\square$

For example, at $n = 5$ the nonzero periodic necklaces form the cycle

$$[00011] \longrightarrow [00101] \longrightarrow [01111] \longrightarrow [00011]. \quad (6.1)$$

Their attached leaves are, respectively, $[00001]$, $[00111]$, and $[01011]$. The other component consists of $[11111] \rightarrow [00000]$ with a loop at $[00000]$. The two graph types are G_1 -20 and G_3 -20. The component masses are 2 and $6 \cdot 5 = 30$, accounting for all 32 words.

6.2 Powers of two: eventual extinction

Over $\mathbb{F}_2$, the middle term in a square vanishes:

$$(I + R)^2 = I + 2R + R^2 = I + R^2.$$

Repeated squaring gives $(I + R)^{2^a} = I + R^{2^a}$ for every nonnegative integer a . Consequently:

Proposition 6.2. *If $n = 2^a$, every word reaches zero after at most n steps. The graph $\mathcal{Q}_n$ has one component, whose maximum transient depth is exactly n .*

Proof. Since $R^n = I$,

$$D^n = (I + R)^n = I + R^n = 0.$$

All necklaces therefore reach $[0]$, proving connectedness. To see that depth n is attained, let e_0 be the word with a single one at position zero. Binary expansion of the exponent gives

$$(I + R)^{n-1} = \prod_{j=0}^{a-1} (I + R^{2^j}) = I + R + \cdots + R^{n-1}.$$

Thus $D^{n-1}e_0 = \mathbf{1} \neq \mathbf{0}$, and the next step is zero. This also covers $n = 1$, using the empty product I . $\square$

In particular, the tree at length 2^a has $2^a + 1$ levels when the root is included. Figure 2 illustrates the case $a = 2$.

Corollary 6.3. *The graph $\mathcal{Q}_n$ is connected if and only if n is a power of two, including $n = 1$.*

Proof. Only the converse remains. If n has an odd divisor $m > 1$, choose a nonzero even-weight word of length m . It is periodic by Proposition 6.1. Repeating it to length n preserves its periodicity and gives a nonzero periodic word. Its necklace lies on a cycle distinct from $[0]$, so there is more than one component. $\square$

7 Reflection is a separate symmetry

Let J reverse a written word. Reversing the order around the circle is different from exchanging zeros and ones. For example, complement changes the weight from k to $n - k$, whereas reflection preserves it.

A necklace is *achiral* if its reflection belongs to the same rotation class, $[Jw] = [w]$. Otherwise it is *chiral* and has a distinct mirror-image necklace. If reflections as well as rotations are identified, the resulting objects are *binary bracelets*, the orbits of a dihedral-group action. Here the vertices remain necklaces, so distinct mirror images remain distinct vertices.

A *palindrome* is a written word equal to its reversal. A palindromic representative guarantees achirality, but the converse fails at even lengths. The necklace $[0101]$ is achiral because its reversal is 1010 , a rotation. Neither of its two written representatives is a palindrome. Geometrically, an axis of reflection may pass through beads as well as through gaps.

Reflection induces an automorphism of the necklace graph. On words,

$$JR = R^{-1}J, \quad JD = R^{-1}DJ.$$

The extra rotation disappears on necklaces, so $\overline{J}\overline{D} = \overline{D}\overline{J}$. Thus reflection maps a component either to itself or to a distinct isomorphic component.

An invariant component need not have every vertex individually fixed by reflection. At length 6, the two distinct chiral necklaces

$$[001011] \quad \text{and} \quad [001101]$$

both map to the achiral necklace $[010111]$. Their reflections are paired within a single component containing both chiral and achiral vertices. The commutation relation implies that an achiral necklace always has an achiral image; it does not imply that all its preimages are achiral.

8 Space–time patterns and cycles with drift

A trajectory can be displayed as an array: write the initial word in row zero, its image in row one, and continue downward. Repeat every row horizontally to display its circular boundary condition. If $a(i, t)$ is the bit at position i and time t , then

$$a(i, t + 1) = a(i, t) + a(i - 1, t) \pmod{2}, \quad a(i + n, t) = a(i, t). \quad (8.1)$$

This is a *space–time diagram* of a one-dimensional additive cellular automaton. Replacing zeros and ones by white and black squares gives a coloring of the square grid obeying the same local rule.

8.1 Pascal’s triangle modulo two

Since I and R commute, the binomial theorem gives

$$D^t = (I + R)^t = \sum_{j=0}^t \binom{t}{j} R^j, \quad (8.2)$$

with coefficients reduced modulo two. Starting from a single one, the nonzero coefficients form Pascal’s triangle modulo two, as in Figure 3. On a circle, contributions whose positions agree modulo n add modulo two and may cancel. The familiar growing triangle describes the initial rows before this circular overlap.

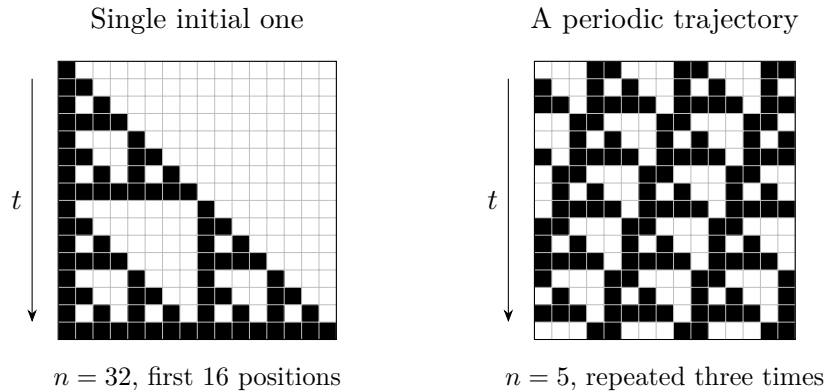


Figure 3: Rows $t = 0, \dots, 15$, with black for 1 and white for 0. Left: the first sixteen rows of Pascal’s triangle modulo two; the omitted sixteen spatial positions are zero in all these rows. Right: evolution from 00011, repeated horizontally. Its necklace has period 3, but the written row first returns at time 15.

8.2 A necklace can return before its written word

Spatial repetition and temporal repetition are different. A necklace on a cycle returns after some least number T of steps, but a chosen word may return rotated:

$$D^T w = R^s w. \quad (8.3)$$

If d is the word’s least spatial period, the *drift* s is well defined modulo d . For example,

$$011 \xrightarrow{D} 110 \xrightarrow{D} 101 \xrightarrow{D} 011.$$

These are three different words representing the same necklace. The necklace is a fixed point, with $T = 1$, even though the word has temporal period 3. Here $d = 3$ and $s = 2$ for our right-rotation convention.

More generally, the least temporal period of the word itself is

$$\tau = T \frac{d}{\gcd(d, s)}. \quad (8.4)$$

To prove this, any return of the word must first be a return of its necklace, so its time is a multiple of T . After kT steps, commutation gives $D^{kT}w = R^{ks}w$. The least positive k with $d \mid ks$ is $d/\gcd(d, s)$. When $s = 0$, the convention $\gcd(d, 0) = d$ gives $\tau = T$.

At $n = 5$, take $w = 00011$. The necklace cycle in (6.1) has length $T = 3$, while

$$D^3(00011) = 00110 = R^4(00011).$$

Thus $d = 5$, $s = 4$, and $\tau = 15$. The right panel of Figure 3 displays these fifteen steps and the returning row.

If the starting word is periodic, extend the array periodically in time as well as space. Equation (8.3) says

$$a(i, t + T) = a(i - s, t), \quad a(i + s, t + T) = a(i, t).$$

The array is invariant under the two translations $(d, 0)$ and (s, T) in the space-time plane. These generate a *period lattice*. A fundamental repeating region can consequently be skew: the necklace cycle need not give a rectangular tile of height T . A rectangular repeat has height τ . This distinction explains how a short cycle of circular patterns can generate a much taller pattern in fixed coordinates.

9 Toward an algebraic classification

The examples suggest three questions for each length n : which graph types occur, how many components have each type, and how many subsets each component represents? Choosing representatives is one way to display the answer, but it is not part of the definition of a partition or a requirement for proving completeness.

The first algebraic step is to encode a word as a polynomial,

$$w_0 w_1 \cdots w_{n-1} \longleftrightarrow w_0 + w_1 x + \cdots + w_{n-1} x^{n-1}.$$

Coefficients lie in $\mathbb{F}_2$, and exponents are taken cyclically, using $x^n = 1$. Thus the word space can be written as

$$V_n = \mathbb{F}_2[x]/(x^n + 1), \quad R w = x w, \quad D w = (1 + x) w. \quad (9.1)$$

The plus sign in $x^n + 1$ is also a minus sign in characteristic two. Rotation and evolution have become multiplication by two fixed polynomials. This representation is part of the established algebraic theory of additive cellular automata and Ducci sequences [3, 2].

Write uniquely

$$n = 2^a m = q m, \quad a \geq 0, \quad m \text{ odd}. \quad (9.2)$$

Here a is the number of factors of two in n , and m is its odd part. For example, $40 = 2^3 \cdot 5$. Repeated squaring gives

$$x^n + 1 = (x^m + 1)^q = (x + 1)^q Q_m(x)^q, \quad Q_m(x) = \frac{x^m + 1}{x + 1}. \quad (9.3)$$

Since m is odd, the factors $x + 1$ and Q_m are relatively prime. The Chinese remainder theorem separates two coordinates:

$$V_n \simeq \underbrace{\mathbb{F}_2[y]/(y^q)}_{\text{transient coordinate}} \oplus \underbrace{\mathbb{F}_2[x]/Q_m(x)^q}_{\text{periodic coordinate}}, \quad y = x + 1.$$

In the first coordinate, D multiplies by y , and $y^q = 0$, so every value eventually vanishes. In the second, $1 + x$ is invertible, so evolution permutes a finite set of values. This is an algebraic explanation of the trees and cycles in the graph.

For a power of two, $m = 1$ and $Q_m = 1$, so the periodic coordinate contains only zero. For odd n , $q = 1$, so the transient coordinate disappears after one step. These are exactly the two situations proved elementarily in Section 6.

The companion paper *The Partitioning of Power Sets Into Graphs* develops this decomposition into a classification of the necklace components. Among its consequences are a common tree depth q , identical level-regular trees at the cycle vertices of a component, and formulas that describe every length $m2^a$ from finite algebraic data associated with the odd part m . Its counting formulas accompany a proof of disjointness and exhaustiveness, accounting for all 2^n subsets.

Necklaces, dynamical components, and graph types organize different levels of structure. A necklace forgets a starting position. A component collects every necklace attracted to one cycle. A graph type retains the arrangement of edges while discarding vertex labels. Keeping these levels distinct allows both the geometry of the dynamics and the exact partition of the power set to be described.

Further reading

Binary necklaces are standard objects in combinatorics on words; [1] provides one entry into their literature. The binary Ducci map and its finite-field formulation are treated in [2], and polynomial methods for additive cellular automata in [3]. Rotation quotients of cellular-automaton graphs, for a broader range of local rules, are considered in [4]. These perspectives place the elementary constructions above within established mathematical frameworks.

References

- [1] C. Degni and A. A. Drisko, *Gray-ordered binary necklaces*, The Electronic Journal of Combinatorics **14** (2007), no. 1, Research Paper 7. doi:10.37236/926.
- [2] M. Misiurewicz, J. G. Stevens, and D. M. Thomas, *Iterations of linear maps over finite fields*, Linear Algebra and its Applications **413** (2006), no. 1, 218–234. doi:10.1016/j.laa.2005.09.002.
- [3] O. Martin, A. M. Odlyzko, and S. Wolfram, *Algebraic properties of cellular automata*, Communications in Mathematical Physics **93** (1984), no. 2, 219–258. doi:10.1007/BF01223745.
- [4] L. Frati, C. Petak, and N. Cheney, *Networks of binary necklaces induced by elementary cellular automata rules*, 2024. arXiv:2409.04780.